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Geometric Construction of AdS Twistors

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Abstract: Time-like geodesics in AdS_4 , AdS_5 and AdS_7 are constructed geometrically and independently of choice of AdS coordinates from division algebra spinors of the corresponding AdS groups, explaining and generalising the construction by Claus *et al.* of AdS_5 twistors.

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1. INTRODUCTION

In connection with the AdS/CFT correspondence [1], there has been a renewed interest in anti-de Sitter (AdS) and sphere (S) geometries, supergravities on $\text{AdS} \times \text{S}$ backgrounds, and perturbative string theory on these spaces. In contrast to flat Minkowski background, where perturbative quantisation of strings and NSR superstrings is straight-forward (and where supersymmetric quantisation of 10-dimensional superstrings recently was achieved [2]), leading to free two-dimensional (super-)conformal field theories, the situation in AdS space looks extremely difficult, in spite of the simple structure of the manifold. The 3-dimensional case, where the bosonic string has been successfully quantised [3], is special in that AdS_3 is a group manifold.

Speculations have occurred whether a twistor formulation might provide a more natural framework for string quantisation on $\text{AdS} \times \text{S}$ spaces [4]. There is not much substantial evidence for this yet, but the idea should be tested. In flat space, twistors are associated with massless particles, since they are spinors under the conformal group. In AdS space, the types of twistors so far considered [5,4,6,7] transform only under the AdS isometry group. Therefore, no particular mass is favoured, and one main argument against the applicability of twistor theory to strings, namely the observation that string theory is not conformally symmetric, is absent. Twistor variables may (at least group-theoretically) generate an entire massive string spectrum.

In spite of some progress, at least to the extent that a superstring action, with clever choice of fermionic coordinates related to Killing spinors [8], now looks simpler than some time ago, quantisation is not yet feasible. The twistor program should be carefully reexamined, and one of the first things to do is to ask what, in general, an AdS twistor is, and what kind of equations it satisfies. This is the subject of the present paper. In the process, a simple geometric definition of the twistor variables, independent of choice of coordinates on AdS space and manifestly covariant under the isometry group, will be obtained.

The sphere parts of the relevant geometries will be left out of the present discussion, which consequently means that we pay no attention whatsoever to physically relevant R-symmetries, nor to Kaluza–Klein spectra. Neither will supersymmetry be considered. We regard this paper as setting the framework for future investigations including these aspects.

2. GEOMETRIC CONSTRUCTION

We start out by demanding that, on-shell, twistors describe the motion of a massive particle on AdS. There is no need to distinguish between AdS and its universal covering space in

this context, since the massive geodesics of the covering space are the (infinite) universal coverings of the AdS geodesics.

Let us regard AdS_{d+1} with radius R as the hyperboloid $\eta_{MN}X^MX^N = -R^2$ in a flat space with signature $(2, d)$. The basic observation on which the construction will be based is the fact that a geodesic on AdS is the intersection between the hyperboloid and a plane, more specifically a completely time-like plane, through the origin $X = 0$. The stability group of a time-like plane is $\text{SO}(2) \times \text{SO}(d)$, so the space \mathcal{M} of massive geodesics on AdS_{d+1} is the homogeneous space

$$\mathcal{M} = \frac{\text{SO}(2, d)}{\text{SO}(2) \times \text{SO}(d)} . \quad (2.1)$$

A plane is the type of geometrical object that may naturally be parametrised as a spinor (*i.e.*, twistor) bilinear. We will now go through this parametrisation, and take it as the operative definition of AdS twistors.

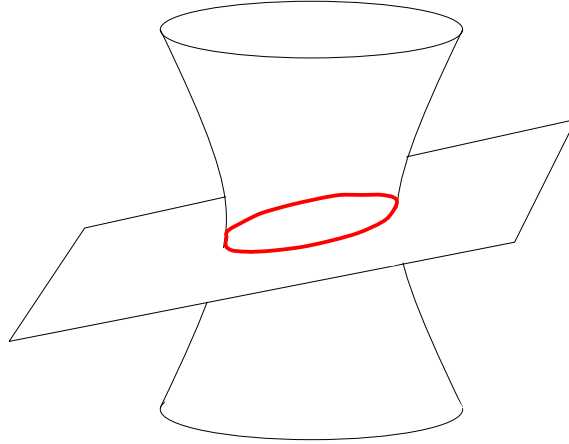


Fig. 1. A geodesic as an intersection with a plane.

When a plane is to be parametrised as a spinor bilinear, the object formed from a spinor Λ that may be used is

$$\Pi^{MN} = \frac{1}{2} \bar{\Lambda} \Gamma^{MN} \Lambda . \quad (2.2)$$

This already gives some group-theoretical information, namely that the symmetric product of two spinor representations of $\text{SO}(2, d)$ must contain the adjoint. All the cases treated here are of this type. In general one may of course consider other situations, where some anti-symmetric/anti-hermitean matrix is introduced in eq. (2.2), but this will at present not be needed.

Eq. (2.2) will be taken as the basic relation between the spinorial and space-time kinematical variables, the "twistor transform". It is interpreted so that the tensor Π defines

the plane in question by spanning directions in the plane. This puts restrictions on the spinor—in general a tensor formed as in eq. (2.2) will not define a single plane. We will find that typically more than one spinor is needed, so that eq. (2.2) must be seen as a sum over some internal orthogonal index carries by Λ (eventually, this index should be a spinor index related to the proper R-symmetry group in an $\text{AdS} \times \text{S}$ setting). A set of spinors fulfilling the adequate constraints to define a plane through eq. (2.2) will be referred to as *simple*. Simple spinors are in a certain sense, fulfilling homogeneous quadratic constraints, quite analogous to pure spinors [9].

We will consider a number of dimensionalities that we find more interesting than a general setting, partly for their mathematical simplicity and partly for their physical relevance. The cases AdS_4 , AdS_5 and AdS_7 are special in that they occur in the supergravity solutions corresponding to the three non-dilatonic branes: the membrane in $D = 11$, the self-dual 3-brane in type IIB in $D = 10$ and the 5-brane in $D = 11$. The corresponding AdS groups $\text{SO}(2, 3)$, $\text{SO}(2, 4)$ and $\text{SO}(2, 6)$ are identical to the conformal groups of 3-, 4- and 6-dimensional Minkowski space, and naturally related to the division algebras \mathbb{R} , \mathbb{C} and \mathbb{H} , the real, complex and quaternionic numbers*. We denote the division algebras by \mathbb{K}_ν , $\nu = 1, 2, 4$, and the AdS groups (or more precisely the spin groups, their double covers) are written as $\text{Sp}(4; \mathbb{K}_\nu)$ in the sense defined by Sudbery [10]. A spinor under the AdS group belongs to the fundamental representation of $\text{Sp}(4; \mathbb{K}_\nu)$, *i.e.*, it is a 4-component column with entries in \mathbb{K}_ν . It transforms in addition under the usual $N = 1$ R-symmetry group \mathbb{Z}_2 , $\text{U}(1)$ or $\text{SU}(2)$ for $\nu = 1, 2$ or 4 respectively, by right multiplication by elements of \mathbb{K}_ν with unit norm. At the algebra level, the R-symmetry is generated by $A_1(\mathbb{K}_\nu)$, imaginary elements of \mathbb{K}_ν , which for higher N generalises to $A_N(\mathbb{K}_\nu)$, anti-hermitean $N \times N$ matrices. Octonions will not be considered in this paper—it is straight-forward to check that non-associativity ruins the corresponding construction for AdS_{11} (to the initiated reader we may state this more precisely: the Moufang identities that allows for the use of S^7 as R-symmetry “group” [11] for Minkowski (2-component) spinors do not suffice here—what is demanded is analogous to the existence of $\mathbb{O}P^3$).

Let us now investigate the possibility that the twistor transform (2.2), when certain constraints are “imposed”, describes the space (2.1) of massive geodesics on $\text{AdS}_{\nu+3}$. To get a better perspective, we would like to compare the present construction to what happens for division algebra twistors in Minkowski space [12, 13]. The geometric construction, which in that case consists of parametrising a light-like vector as a spinor bilinear, then only involves momenta, and not the coordinates: $p^\mu = \bar{\lambda} \gamma^\mu \lambda$. The momentum is invariant under R-symmetry, so this is simply divided out from spinor space to obtain the configuration space

* We are not aware of any reason for this intriguing coincidence of physical and mathematical structures—the mathematical sequence is natural, but the three M-theory vacua do not share a corresponding common origin.

of momenta. This modding out is the Hopf map $S^{2\nu-1} \xrightarrow{S^{\nu-1}} S^\nu$. In the present case, the twistor transform involves the entire phase space^{*}, and the spinor Λ will be self-conjugate. The symplectic form on twistor phase space is essentially unique, given the AdS symmetry—it must equal the antisymmetric (or anti-hermitean) spinor metric[†], $\{\Lambda^A, \Lambda^B\} = \varepsilon^{AB}$. Therefore, we do not need to derive it from the Poisson brackets of coordinates and momenta for some specific choice of AdS coordinates. The space \mathcal{M} of geodesics will not be constructed simply as the quotient of a spinor space by a symmetry group, but as such a quotient of a constraint hypersurface in spinor (this is true whether or not there are second class constraints—if there is, the dimension of the gauge orbit will be smaller than the codimension of the constraint surface. We will soon see that second class constraints generically *are* present, with the case of AdS₅ as an exception).

3. THE TWISTOR TRANSFORM FOR ADS₄

We must determine how many spinors are needed in order to form a time-like plane, *i.e.*, how to obtain a simple spinor. A straight-forward and practical, but not very elegant, way of doing this is by using the AdS group to choose a specific frame in which the antisymmetric tensor Π defining the plane takes a particularly simple form. We let it be $\Pi^{00'} \neq 0$ and other components zero, where 0 and 0' denote the two time directions. We refer to the Appendix for further notation. For simplicity, we will do the 4-dimensional (real) case explicitly, and then comment on the details of AdS₅ and AdS₇. Using the Γ -matrices of the appendix, a number of equations are obtained that describes the vanishing of all components of Π^{MN} except $MN = 00'$. It is straight-forward to verify that the minimum number N of spinors needed is $N = 4$. The four spinors are conveniently collected in a quaternion, so that

$$\Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \in \mathbb{R}^4 \otimes \mathbb{H} = \mathbb{H}^4 . \quad (3.1)$$

The R-symmetry for such spinors is $SU(2)_L \times SU(2)_R$, acting by left and right quaternionic multiplication, which obviously commutes with left multiplication by real 4×4 matrices. In

^{*} Such an interpretation may be given also in Minkowski space, if Π in eq. (2.2) is taken to contain the generators of conformal transformations. All the interesting geometry lies in the parametrisation of the momentum, however, since the space of geodesics is obtained by a direct product with a Cauchy surface.

[†] This equation of course refers to brackets between *real* components, so in the complex and quaternionic cases the spinor index is not identical to that of the Appendix, although the same letters are used.

order for $\bar{\Lambda}\Gamma^{MN}\Lambda$ to span the $00'$ plane, the algebraic constraints on the λ_i 's is that they are all orthogonal as quaternions (SO(4) vectors) and of equal length:

$$\begin{aligned} \text{Re}(\bar{\lambda}_a\lambda_b) &= 0, \quad a \neq b \\ |\lambda_a| &= |\lambda_b|. \end{aligned} \tag{3.2}$$

This is satisfied iff *either*

$$\begin{aligned} T_{(L)i} &\equiv \frac{1}{2}\text{Re}(\bar{\Lambda}e_i\Lambda) = 0 \\ \text{or } T_{(R)i'} &\equiv -\frac{1}{2}\text{Re}(\bar{\Lambda}\Lambda e_{i'}) = 0, \end{aligned} \tag{3.3}$$

where $T_{(L,R)}$ are the generators of the R-symmetry SU(2)'s. This formulation of the constraints is covariant, and must therefore hold in any frame, not just the one chosen for the calculation. The two alternatives are mutually exclusive (a spinor fulfilling both has to vanish), and correspond to the λ_a 's spanning an orthogonal frame of either orientation.

There is now a problem: only part of the R-symmetry generators (half) act as algebraic constraints on the spinors, but also the rest need to be eliminated in order to remove the redundancy of the simple spinor description of the plane. Let us arbitrarily choose $T_{(L)} = 0$. We then also need a set of algebraic constraints generating SU(2)_R, but without setting $T_{(R)} = 0$. The only possibility is $T_{(R)} = v$, where v is some non-zero imaginary quaternion. Specifying v implies that SU(2)_R is broken to U(1). The unbroken U(1) constraint is 1st class and the remaining two constraints are 2nd class. The physical significance of v is examined in the following section.

We may now check the construction by counting the physical degrees of freedom. The spinor Λ had from the outset 16 real components. With two 2nd class constraints and four 1st class ones (generating SU(2)×U(1)), we arrive at $16 - 2 - 2 \times 4 = 6$ degrees of freedom. This should be checked against the dimensionality of the space of massive geodesics $\mathcal{M} = \text{SO}(2,3)/(\text{SO}(2) \times \text{SO}(3))$, which is $10 - 1 - 3 = 6$.

It is of course a bit disappointing that 2nd class constraints are present. Although they may be eliminated using the Dirac procedure without breaking AdS covariance, the resulting Dirac brackets are Λ -dependent and do not invite to quantisation.

4. $D = 5$ AND 7

It is a straight-forward exercise to continue the analysis to AdS₅ and AdS₇. It is already known [5] that the complex (AdS₅) construction works with two spinors. In the quaternionic

case, four spinors are again needed. This non-uniform pattern is somewhat surprising—nothing corresponding happens for Minkowski space twistors (where also the octonionic case works [14,13]). Of course, the AdS_5 construction also works with four spinors, but this is a redundant description. We will comment on this below.

A pair of complex spinors has the R-symmetry group $A_2(\mathbb{C}) \approx \text{SU}(2) \times \text{U}(1)$. At the same time, the condition that $\Pi^{MN} = \frac{1}{2}\bar{\Lambda}\Gamma^{MN}\Lambda$ spans a plane is identical to the vanishing of the generators of the $\text{SU}(2)$ R-symmetry subgroup, $T_i = 0$. The $\text{U}(1)$ generator is not allowed to vanish, but is taken to fulfill an inhomogeneous constraint $T = v$. Since T generates an abelian subgroup, the set of four constraints is nevertheless 1st class, unlike the real case, where an inhomogeneous constraint had to be imposed on a non-abelian part of the R-symmetry generators. The dimensionalities clearly match: $\dim \mathcal{M} = \dim(\text{SO}(2,4)/(\text{SO}(2) \times \text{SO}(4))) = 8$, while the number of physical twistor degrees of freedom is $16 - 2 \times 4 = 8$.

If one choses to use four spinors in the parametrisation of AdS_5 geodesics, the R-symmetry is $A_4(\mathbb{C}) \approx \text{SU}(4) \times \text{U}(1)$. The entire set of $\text{SU}(4)$ generators can not vanish. The R-symmetry is then broken to $\text{SU}(2) \times \text{SU}(2) \times \text{U}(1) \times \text{U}(1)$, with the rest being 2nd class constraints. Without going into explicit detail, the two-spinor case can be recovered by using the eight 2nd class constraints and simultaneously gauge fixing one of the $\text{SU}(2) \times \text{U}(1)$ factors, thereby eliminating two of the spinors.

For the quaternionic twistors of AdS_7 , four spinors are again needed. The R-symmetry is $A_4(\mathbb{H}) \approx \text{Sp}(8)$. Again, only part of the generators may fulfill homogeneous constraints, which means that there will be a mixture of 1st and 2nd class constraints. Here, the condition for $\Pi^{MN} = \frac{1}{2}\bar{\Lambda}\Gamma^{MN}\Lambda$ to span a plane implies that $\text{Sp}(8)$ has to be broken to $\text{SU}(4) \times \text{U}(1)$, of which the $\text{U}(1)$ constraint, as usual, is inhomogeneous. Counting the twistor degrees of freedom from the constraint structure gives $64 - 2 \times 16 - 20 = 12$, matching the dimension of $\mathcal{M} = \text{SO}(2,6)/(\text{SO}(2) \times \text{SO}(6))$.

What is the meaning of v , the length of the non-vanishing R-symmetry generator? The twistor transform (2.2), that described algebraically the plane in the flat embedding space, also contains the generators of the AdS_{d+1} group $\text{Spin}(2,d)$. The actual geodesic described by the twistors is independent of the scaling of Λ , *i.e.*, of v . In this way the choice of v gives the only piece of information about particle motion not contained in the geodesic, namely the mass. The exact relation is easily derived in the frame used above, where $|\Pi^{00'}| = 2|v|$. $\frac{1}{2}\Pi^{00'}$ is the generator of translations in the time variable φ , the angle about the “equator” of AdS , so the metrically normalised translation generator is $\frac{1}{2R}\Pi^{00'}$. We therefore obtain the relation

$$m = \frac{|v|}{R} \quad (4.1)$$

(the numerical factor of course depending on the normalisation of the corresponding Cartan subalgebra element; the factor in eq. (4.1) relates to the explicit formulæ in section 3).

5. DISCUSSION

We have given an explicit and geometric division algebra twistor transform for the spaces of massive geodesics on the three AdS spaces corresponding to non-dilatonic branes.

It is noteworthy that the previously known construction of twistors for geodesic motion on AdS₅ has a more elegant structure than those for AdS₄ and AdS₇. The latter two are probably not very useful, since they contain second class constraints that make the twistor descriptions, independently of any actual dynamics, if not intractable, so at least quite non-linear. They will probably not present any advantages over the coset description. It is difficult to refrain from speculating that the particular properties of AdS₅ twistors might have something to do with AdS₅ arising from string theory in $D = 10$, while the other two spaces derive from eleven-dimensional supergravity/M-theory, although the mathematical reason is simply that the R-symmetry group in the complex case, and only then, contains a U(1) factor.

It is of course straight-forward to write down 1st-order actions that reproduce the particle dynamics:

$$S \sim \int d\tau \left[(\bar{\Lambda} \dot{\Lambda}) + \varrho \cdot (\bar{\Lambda} \Lambda (T - v)) \right] , \quad (5.1)$$

where ϱ are Lagrange multipliers.

Apart from the possible application to string quantisation, the procedure of which is not obvious, there are two ways the present formalism has to be extended. The supersymmetric case must be addressed, which can probably be done along the same lines as for supertwistors in Minkowski space, and one will have to consider geodesic motion on the accompanying sphere manifolds. Hopefully, the present formalism will extend naturally to a situation where the R-symmetry acts as isometries of the spheres, and where the twistors, suitably constrained, simultaneously parametrise planes in the embedding spaces of both anti-de Sitter space and the sphere. The constraint structure will certainly be affected when the R-symmetry group (or part of it) no longer is gauged, and it is not clear that the structure found in this paper, with a mixing of 1st and 2nd class constraints, will persist. Whether or not the methods introduced here can be useful for the quantum-mechanical treatment of string theory on AdS spaces remains to be investigated. The geometric approach can hopefully help to gain insight in this question.

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REFERENCES

- [1] J. Maldacena, “*The large N limit of superconformal field theories and supergravity*”,
Int. J. Theor. Phys. **38** (1999) 1113 [[hep-th/9711200](#)];
- S.S. Gubser, I.R. Klebanov and A.M. Polyakov,
“*Gauge theory correlators from non-critical string theory*”,
Phys. Lett. **B428** (1998) 105 [[hep-th/9802109](#)];
- E. Witten, “*Anti de Sitter space and holography*”,
Adv. Theor. Math. Phys. **2** (1998) 253 [[hep-th/9802150](#)].
- [2] N. Berkovits, “*Super-Poincaré covariant quantization of the superstring*”, [hep-th/0001035](#).
- [3] J. Maldacena and H. Ooguri, “*Strings in $AdS(3)$ and $SL(2,R)$ WZW model. 1.*”, [hep-th/0001053](#).
- [4] P. Claus, R. Kallosh and J. Rahmfeld, “*BRST quantization of a particle in AdS_5* ”, [hep-th/9906195](#).
- [5] P. Claus, M. Günaydin, R. Kallosh, J. Rahmfeld and Y. Zunger, “*Supertwistors as quarks of $SU(2,2|4)$* ”,
J. High Energy Phys. **9905** (1999) 019 [[hep-th/9905112](#)].
- [6] I. Bandos, J. Lukierski, C. Preitschopf and D. Sorokin,
“ *OSp supergroup manifolds, superparticles and supertwistors*”, [hep-th/9907113](#).
- [7] Y. Zunger, “*Twistors and Actions on Coset Manifolds*”, [hep-th/0001072](#)
- [8] R. Kallosh and A.A. Tseytlin, “*Simplifying superstring action on $AdS_5 \times S^5$* ”,
J. High Energy Phys. **9810** (1998) 016 [[hep-th/9808088](#)].
- [9] M. Cederwall, “*Bi-division algebra spinors in $D=10$ Minkowski geometry*”,
Göteborg Preprint 91-26 (1991).
- [10] A. Sudbery, “*Division algebras, (pseudo)orthogonal groups and spinors*”, J. Phys. **A17** (1984) 939.
- [11] M. Cederwall and C.R. Preitschopf, “ *S^7 and $\widehat{S^7}$* ”,
Commun. Math. Phys. **167** (1995) 373 [[hep-th/9309030](#)].
- [12] I. Bengtsson and M. Cederwall, “*Particles, twistors and the division algebras*”,
Nucl. Phys. **B302** (1988) 81.
- [13] M. Cederwall, “*Octonionic particles and the S^7 symmetry*”, J. Math. Phys. **33** (1992) 388.
- [14] N. Berkovits, “*A supertwistor description of the massless superparticle in ten-dimensional superspace*”,
Nucl. Phys. **B350** (1991) 193.

APPENDIX A: SPINORS AND GAMMA MATRICES

Here, we set the conventions for the gamma matrices used in the twistor transform (2.2). They can be given in a unified notation for the three cases. We denote by e_I , $I = 1, \dots, \nu$, the standard orthonormal basis for the division algebra \mathbb{K}_ν . For the flat embedding space with signature $(2, \nu + 2)$ we use light-cone coordinates $M = (\oplus, \ominus, \mu) = (\oplus, \ominus, +, -, I)$ and scalar product $V \cdot W = -V^\oplus W^\ominus - V^\ominus W^\oplus - V^+ W^- - V^- W^+ + V^I W^I$.

A spinor under the AdS group belongs to the fundamental representation of $\text{Sp}(4; \mathbb{K}_\nu)$, *i.e.*, it is a 4-component column with entries in \mathbb{K}_ν . Using a dotted/undotted notation for spinors, and in addition primed and unprimed spinor indices (since there generically are two chiralities), the gamma matrices (or, strictly speaking, sigma matrices) acting on one chirality (the unprimed one that is chosen for the twistors) are

$$\begin{aligned} \Gamma^{\oplus A'}_B &= \begin{bmatrix} \sqrt{2} \mathbb{1}^\alpha_\beta & 0 \\ 0 & 0 \end{bmatrix}, & \Gamma^{\ominus A'}_B &= \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \mathbb{1}_{\dot{\alpha}\dot{\beta}} \end{bmatrix}, \\ \Gamma^{\mu A'}_B &= \begin{bmatrix} 0 & \tilde{\gamma}^{\mu\alpha\dot{\beta}} \\ \gamma^\mu_{\dot{\alpha}\beta} & 0 \end{bmatrix}, \end{aligned} \quad (\text{A.1})$$

and on the other one

$$\begin{aligned} \tilde{\Gamma}^{\oplus A}_{B'} &= \begin{bmatrix} 0 & 0 \\ 0 & -\sqrt{2} \mathbb{1}_{\dot{\alpha}\dot{\beta}} \end{bmatrix}, & \tilde{\Gamma}^{\ominus A}_{B'} &= \begin{bmatrix} -\sqrt{2} \mathbb{1}^\alpha_\beta & 0 \\ 0 & 0 \end{bmatrix}, \\ \tilde{\Gamma}^{\mu A}_{B'} &= \begin{bmatrix} 0 & \tilde{\gamma}^{\mu\alpha\dot{\beta}} \\ \gamma^\mu_{\dot{\alpha}\beta} & 0 \end{bmatrix}, \end{aligned} \quad (\text{A.2})$$

where $\gamma^\mu, \tilde{\gamma}^\mu$ are $\text{SL}(2; \mathbb{K}_\nu) \approx \text{Spin}(1, \nu + 1)$ gamma matrices:

$$\begin{aligned} \gamma^+_{\dot{\alpha}\beta} &= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}, & \gamma^-_{\dot{\alpha}\beta} &= \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{bmatrix}, & \gamma^I_{\dot{\alpha}\beta} &= \begin{bmatrix} 0 & \bar{e}_I \\ e_I & 0 \end{bmatrix} \\ \tilde{\gamma}^{+\alpha\dot{\beta}} &= \begin{bmatrix} 0 & 0 \\ 0 & -\sqrt{2} \end{bmatrix}, & \tilde{\gamma}^{-\alpha\dot{\beta}} &= \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}, & \tilde{\gamma}^{I\alpha\dot{\beta}} &= \begin{bmatrix} 0 & \bar{e}_I \\ e_I & 0 \end{bmatrix}. \end{aligned} \quad (\text{A.3})$$

The matrices Γ^{MN} used in the construction of the plane defining the geodesics are constructed as

$$\Gamma^{MNA}_B = \frac{1}{2}(\tilde{\Gamma}^M \Gamma^N - \tilde{\Gamma}^N \Gamma^M)^A_B, \quad (\text{A.4})$$

and the twistor bilinear is given by

$$\Pi^{MN} = \frac{1}{2} \bar{\Lambda} \Gamma^{MN} \Lambda \equiv \frac{1}{2} \bar{\Lambda}_A \Gamma^{MNA}_B \Lambda^B \equiv \frac{1}{2} \Lambda^{\dagger \dot{A}} \epsilon_{\dot{A}B} \Gamma^{MNB}_C \Lambda^C, \quad (\text{A.5})$$

where the anti-hermitean “spinor metric”

$$\varepsilon_{\dot{A}B} = \begin{bmatrix} 0 & \mathbb{1}_{\dot{\alpha}}^{\dot{\beta}} \\ -\mathbb{1}_{\alpha\beta} & 0 \end{bmatrix} \quad (A.6)$$

is used to lower the spinor index, and where \dagger implies division algebra conjugation. In the real case, dots are of course superfluous, and there is only one chirality. The above formulæ are still correct, and primed and unprimed indices are then related via

$$E^{A'}_{B} = \begin{bmatrix} 0 & \varepsilon^{\alpha\beta} \\ \varepsilon_{\alpha\beta} & 0 \end{bmatrix} . \quad (A.7)$$